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# A LETTER TO RICHARD PRICE, D.D. and F.R.S.

CONTAINING

AN ENTIRE REFUTATION

OF HIS CELEBRATED

TREATISE OF OBSERVATIONS

ON

REVERSIONARY PAYMENTS,

SCHEMES FOR PROVIDING

ANNUITIES FOR WIDOWS,

AND

FOR PERSONS IN OLD AGE.

By SAMUEL CLARK.

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To Dr. RICHARD PRICE, F. R. S.

*Reverend Sir,*

MALEVOLENCE and detraction I ever held in the utmost detestation; and, notwithstanding an attack upon your character as a mathematical writer, although conducted with decent propriety, may be censured as bordering upon either of those passions, I am not conscious to myself of being actuated by any such pernicious motives. The chief view I had in writing this pamphlet was, to communicate to the public such general observations as I have made upon the successive editions of Dr. PRICE's Treatise of REVERSIONARY PAYMENTS, and which, I apprehend, will enable the reader of that celebrated work to form an

adequate judgment of the truth or falſhood of the various articles contained therein. Another, and indeed not leſs cogent reaſon than the former, alſo ſtimulated me to the enquiry; which was, a ſincere deſire to clear the poſthumous reputation of thoſe late eminent mathematicians Meſſrs. SIMPSON and DEMOIVRE, from ſome very injuſt reflections thrown upon their writings in ſeveral pages of your performance abovementioned. Whether I may have ſucceeded in this attempt, muſt be left to the determination of the reader; yet yourſelf will undoubtedly allow the deſign to be truly laudable, more eſpecially as thoſe injured perſons are now not living to defend themſelves, and whoſe errors, (if ſuch they had) you have by this means protracted beyond the limits of the grave. It has always been deemed ungenerous to traduce the memory of departed men; and if we cannot tranſmit their names to poſterity with conſiſtent advantage, it were ſurely far more honourable to conſign them to oblivion. But how injurious to their fame will your treatment

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of thoſe illuſtrious mathematicians appear, ſhould it happen to be proved, that all you have advanced againſt them is entirely deſtitute of even the leaſt foundation of truth? I ſay, ſhould this happen, will you not, for your own ſake, then ſincerely wiſh you had omitted that ſevere cenſure which is now but too apparent throughout your whole treatiſe? If, indeed, you have in ſome few places praiſed Mr. Simpſon, you have almoſt immediately cancelled that ſervice, by adverting to ſome imaginary miſtakes. Thus at page 247 you tell us, " The ingenious and accurate Mr. Simpſon ſaw that it was neceſſary to correct the London Tables, and he has done it with great judgment." This indubitably is a compliment paid to the memory of that eminent mathematician; but you then add, " he has (in your opinion) corrected thoſe tables too imperfectly, and without going upon any fixed principles, or ſhewing particularly how tables of obſervation ought to be formed, and how far in different circumſtances, and at different ages, they are to be depended on." Does not the lat-

ter part of this extract entirely destroy the force of the good offices intended by the former? or can it be supposed you were sincere in that praise which you instantly defeat by the severity of your censure? Had Dr. Price's treatise been of the most perfect kind imaginable, it certainly could have received no additional recommendation, by being contrasted with the imperfect writings of other men. Truth needs no support, and however the diamond may be said to shine forth with more distinguished lustre when surrounded by inferior gems, yet surely a mathematical performance, written by such a celebrated author as yourself, must, in the opinion of the whole world, have risen far superior to any other of the same kind, by the mere force of native excellence alone.

It is now scarce six years since the first publication of these OBSERVATIONS on REVERSIONARY PAYMENTS, and the success thereof has been such as to occasion a farther demand of two other additions, within the space of little more than three years. This, I must confess, if

if we may judge of the merit of a work by the rapidity of ſale, appears greatly in favour of it, and ſeems farther confirmed by the almoſt general approbation with which it has hitherto been received. I would here be underſtood to mean, with reſpect to the mathematical articles, the other parts, which are more of a political nature, have indeed been ſufficiently controverted, particularly where you treat of national debt, ſinking fund, enormous ſums of money to be raiſed by compound intereſt, &c. An example of the latter you give us at page xiii. of the preface, in a note marked *(a)* where it is ſaid, that " a penny put out to 5 *per cent*. compound intereſt at our Saviour's birth, would by this time have increaſed to more money than would be contained in 150 millions of globes, each equal to the earth in magnitude, and all ſolid gold." The *Critical Reviewers* in their obſervations upon your treatiſe, called *An appeal to the public on the ſubject of the national debt*, took notice of the abſurdity in your manner of applying this property of numbers,

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which indeed, conſidered in an abſtracted ſenſe, is moſt undoubtedly true; that is, if 1.05 be multiplied 1777 times into itſelf, the laſt product divided by 240 times .05, and the quotient called pounds may be equal, or perhaps exceed the value of a quantity of gold 150 millions of times greater than the magnitude of the earth; but there is a wide difference between the multiplication of numbers and the multiplication of gold, nor can all the intereſt in the univerſe ever realize one ſhilling of ſpecie, it only transfers property from one hand to another; and however calculations of this kind may appear feaſible upon paper, we cannot help (ſay the Reviewers) being of opinion, that any man of plain common underſtanding would ſmile to hear a mathematician talk of actually raiſing a ſum of gold by compound intereſt, much greater in magnitude than the whole earth itſelf." Waving at preſent this imaginary ſcheme for accumulating riches, I ſhall now proceed to an examination of thoſe articles in your treatiſe on *Reverſionary Payments*, which I apprehend

either

either ſtand in need of farther illſturation, or are in themſelves defective. The firſt queſtion in your treatiſe I take to be of the latter kind; it is to this effect. A ſet of married men enter into a ſociety for ſecuring annuities to their widows: What ſum of money, in a ſingle preſent payment, ought every member to contribute, in order to entitle his widow to a certain annuity for her life, eſtimating intereſt at a given rate *per cent*.

Your method of anſwering this queſtion is, to ſubtract the value of an annuity on the joint continuance of any two lives, from the value of an annuity on the life in expectation, the remainder gives the true value of an annuity on what may happen to remain of the latter of the two lives after the other.

The two lives are ſuppoſed equal in your ſolution, and the anſwer is certainly true with regard to a man's purchaſing an annuity for his wife to commence after her deceaſe; but I apprehend that ſome conſiderable advantage will accrue to an inſtitution of this kind, proportional to the number of members of which the ſociety may occaſionally happen to conſiſt;

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of this you ſeem to take no notice in your inveſtigation. In your anſwer to the ſecond queſtion, page 3, you obſerve, as every marriage will produce either a widow, or widower, it muſt follow, that if every man and his wife were of the ſame age, and the chance equal which ſhall die firſt, the number of widows that have ever exiſted in the world would be equal to half the number of marriages. If you ſpeak of this as what muſt have happened, you are miſtaken, there can only be a certain degree of probability that this might have been the caſe. You next tell us, that, "if every other perſon in ſuch a ſociety (deſcribed in the firſt queſtion) leaves a widow, there muſt ariſe from it a number of widows equal to half its own number;" moſt undoubtedly true; but is it a certainty, that every other man will die before his wife? The chance I grant is the ſame which ſhall firſt deceaſe, because their ages are ſuppoſed equal; but even this, as you juſtly remark of the former, does not determine what number, all living at one and the ſame time, the ſociety may expect will come to be

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conſtantly upon it. In order to determine this particular requiſite in the queſtion, you firſt obſerve, that "if every widow lived no more than a year, the ſociety would never have more annuitants upon it than come on in a year." Granted; and again, "if none of the widows ever died, the number of annuitants would encreaſe for ever." This is granted alſo. From theſe poſtulata, aſſiſted by the help of a fluxionary calculus, you determine the duration of ſurvivorſhip between perſons (married ones) of the ſame ages, to be equal to the duration of marriage; and either of theſe your readers are informed is exactly one third of the common complement of life. This concluſion involves a difficulty not eaſily removed; for the expectation of life, due to an individual of a certain age, is one half of the complement thereof, and conſequently a man or woman of the ſame age may expect a longer ſhare of life, by entering into the marriage ſtate, than if each had remained ſingle. To illuſtrate this, ſuppoſe two perſons, a man and woman, each 32 years of age, their complement of life 54

years, this premiſed, it follows, by a well-known hypotheſis, that the ſhare of life, or number of years, due to either of thoſe ages, is exactly 27, that is, either of them ſingly may juſtly expect to arrive at the age of 59 years. Now according to your determinations, theſe perſons in a marriage ſtate may reaſonably expect to continue therein 18 years; and as the ſurvivorſhip muſt, according to your doctrine, continue the ſame time, it is very clear that either party may expect to live to the age of 68 years; whereas, had they continued in a ſtate of celibacy, the expectation of life due to either of them would not have exceeded 59 years.

I purpoſely paſs over the 3d, 4th and 5th queſtions; for as their ſolutions depend upon the ſame principles with the former, they are certainly obnoxious to ſimilar objections.

The ſolution to your 6th queſtion is indiſputably right, nor do I object to the ſolutions of the 7th and 8th; it is true, the anſwer to the 8th is founded upon a miſtaken hypotheſis, as I have already remarked; but as you have

reaſoned

reaſoned right upon that hypotheſis, I take no farther notice of it.

At page 31, you enquire into what money in hand ought a perſon of an aſſigned age to give for a ſum of money, payable at his death to his heirs? and to anſwer this queſtion you propoſe thus: Subtract the value of the life from the perpetuity; multiply the remainder by the product of the given ſum into the intereſt of 100l. for a year, and this laſt product divide by 100l. encreaſed by its intereſt for a year, will give the anſwer required. In the example you ſuppoſe the life 30, the ſum 100l. and the rate of intereſt 4 *per cent*. The anſwer becomes 39l. 14s. which indeed differs very little from the truth; and had you omitted the diviſion, your ſolution would have been juſt. I do not take notice of this circumſtance as an error in your judgment, but rather impute it to a notion you had entertained concerning ſome ſort of difference between reverſionary annuities and reverſionary ſums; this appears frequently in your treatiſe (by way of *jeu de eſprit* I imagine) particularly in note (E) where the reader is in-

formed, that 25l. is the present value of an estate of 1l. per annum for ever, interest being at 4 *per cent*. that is, it is the value of it, supposing the first rent is to be paid a year hence. If the first rent is to be received immediately, it is worth one year's purchase more, or equivalent to 26l. If Dr. Price is really serious in this admonition, and upon which he grounds a suggestion of a necessary correction to be applied to the common solutions of several important problems in Mr. Simpson's treatise on life annuities, it may be worth while to continue the enquiry a little farther. I mean with respect to the utility of such corrections. As to the truth of the assertion, it is most obvious, namely, that if I chance to receive one pound above my due, I then have one pound more than I ought to have, no doubt of it. But why must this be applied to reversionary payments? Is it usual in those cases to receive 26l. when only 25l. is due, or 260l. for the right of 250? Indeed, whenever this shall happen, the proposed correction should, in the strictest justice, take place. But, certainly, in negociating any

sort

ſort of buſineſs relating to annuities upon Joint or ſingle lives, reverſions, &c. in the manner it is now generally underſtood. Your obſervations abovementioned are entirely unconcerned. However, as you appear deſirous (page 31) to inforce this requiſition, and for ſuch purpoſe have given a demonſtration (in note E) of its uſefulneſs in computations of this kind, it may not be amiſs to conſider how it is to be underſtood. The firſt part $S \times \overline{\frac{1}{nr} + \frac{1}{nr^2} + \frac{1}{nr^3}}$ &c. $(n)$ admitting your correction neceſſary, I make no objection to, the other I believe is defective; indeed it ſeems difficult to underſtand what $\overline{1 - \frac{n-1}{n}} \times \frac{1}{r} + \overline{1 - \frac{n-2}{n}} \times \frac{1}{r^2} + \overline{1 - \frac{r-3}{n}} \times \frac{1}{r^3}$ &c $(n) + \frac{1}{r^{n+1}} + \frac{1}{r^{n+2}}$ &c. can poſſibly mean. You tell us, it is equivalent to $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}$ &c. $- \frac{n-1}{nr} + \frac{n-2}{nr^4} + \frac{n-3}{nr3}$ &c. but this does not ſeem to be true, even if we ſuppoſe $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}$ &c. to be continued in infinitum, concerning which you leave us entirely at a loſs. Here certainly muſt have been ſome miſtake made; for the reſult of the algebraical proceſs will not terminate in the manner you ſuppoſe, i. e. "The value of the life ſubtracted from the perpetuity." Previous to this concluſion you

you reaſon thus : " The value of 1l. to be received at the end of a year, provided the life, whoſe complement is $n$, fails within that time, is the probability of the failure of the life multiplied by 1 l. diſcounted for a year, or $\overline{1 - \frac{n-1}{n}} \times \frac{1}{r}$." This is undoubtedly true. " In the like manner, the value of 1l. to be received at the end of two years, if the ſame life fails in two years, is $\overline{1 - \frac{n-2}{n}} \times \frac{1}{r^2}$." This is likewiſe true ; but then it certainly includes the former ; for a life being ſuppoſed to fail in two years, as here expreſſed, does not reſtrict the event to the ſecond of thoſe years . It might have happened in the firſt, but this was before brought into conſideration, and therefore it ſhould now be excluded from the ſecond term in the required probability, and the ſame ſort of excluſion ſhould have taken place in the remaining years.

The queſtion here adverted to, viz. the 10th in your treatiſe, you ſay, " is the ſame with the 16th (it ſhould be the 26th) problem in Mr. Demoivre's Treatiſe on Annuities, and problem 26th in Mr. Simpſon's Select Exercises," but the anſwers there, you add, are

are materially wrong, when applied to reversionary sums.—How far you may be right with regard to this assertion, I will not pretend to determine. I must, however, take the liberty to observe, that Simpson and Demoivre cannot both be wrong, for this very obvious reason: Demoivre's rule for solution in the case of a reversionary annuity, and Simpson's for an equivalent reversionary sum, exactly coincide; and therefore you will pardon me, should I incline to think neither of these authors are mistaken; be this as it may, I beg leave to propose as follows.

What is the present worth of a certain sum to be received immediately upon the decease of B, who is now of a given age?

*Solution*. Let $n$ be the complement of the given life. If B dies within the first year beyond his present age, of which the probability is $\frac{1}{n}$ the sum becomes due without any discount, consequently the expectation of the legatee thereon is $\frac{S}{n}$ (S being the legacy.) Again, the present worth of his expectation upon receiving the sum S before the end of two years, is compounded of the probability of his living

beyond

beyond the end of the first year: dying in the second, and the discount of the sum S for one year, that is $\frac{S}{r}\times\frac{n-1}{n}\times\overline{1-\frac{n-2}{n-1}}$ or $\frac{S}{rn}$ in the same manner the present worth of the several expectations upon the contingent circumstance of the legacy being received before the end of the 3d, 4th, 5th, &c. years, will be found equal to $\frac{S}{nr^2}\ \frac{S}{nr^3}\ \frac{S}{nr^4}$ &c. respectively, and therefore $\frac{S}{n}$ into $1+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}$ &c. continued to $n$ terms, is the present worth of the legacy to be received at the demise of B. For the sum of this series put $z$, then we have $1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$——$\frac{1}{r^{n-1}}=z$, and $\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$——$\frac{1}{r^{n-1}}+\frac{1}{r^n}=\frac{z}{r}$ subtract the latter from the former, there remains $1-\frac{1}{rn}=z-\frac{z}{r}$ consequently $z=\frac{r-\frac{1}{r^{n-1}}}{r-1}$ and $\frac{S}{n}\times z=\frac{S}{n}\times\frac{r-\frac{1}{r^{n-1}}}{r-1}$ But it is well known that the present worth of an annuity of 1l. to continue during a life, whose complement is $n$ years, is defined by $\frac{1}{r-1}-\frac{r-\frac{1}{r^{n-1}}}{\overline{r-1}^2\times n}$ whence it is evident, that (because $\frac{1}{r-1}$ is the perpetuity,) The perpetuity *minus* the proposed life, being multiplied by $\overline{r-1}\times S$, gives the required answer; hence

hence this rule. Subtract the value of the proposed life from the perpetuity, then it will be as the perpetuity is to the remainder; so is the proposed sum (as a legacy) to its value in present money. This theorem is the same with Simpson's *Select Exercises*, p. 293, for the solution of the 26th problem which relates to a reversionary sum of 500l. to be received on the decease of a person whose present age is 32 years, interest 4 *per cent.* and consequently the present worth thereof just 246l. This reversionary sum of 500l. is most indubitably the same with an estate in perpetuity of 20l. per annum. Now according to Demoivre, the question may run thus. A is to have an annuity of 20l. for him and his heirs after the failing of a given life, which we suppose to be worth 12.7 years purchase, required the present value of his expectation. To answer this, Demoivre's rule is, multiply the value of the given life by the interest of 1l. then subtract that product from unity, let the remainder be multiplied by the value of the perpetuity, and again by the annuity, this last product will be

the value required, and is here equal to 246l. Does not all this make directly against what you have already advanced, namely, that Demoivre and Simpson are right only in the case of reversionary estates? for here you see, that Demoivre, according to your own hypothesis, is right, and therefore no correction is required. Again, Simpson's solution is the same, though applied to a reversionary sum, and yet you assert it should be corrected; it must certainly (I think) follow, that their solutions, applied either to reversionary sums or annuities, need no correction whatsoever.

At page 48 you have this example.—An estate, or an annuity of 10l. for ever, will be lost to the heirs of a person now 34, should his life fail in 11 years. What ought he to give for the assurance of it for this term?

For the demonstration of the method of solution you here use, and by which you determine the value of the assurance to be 43.4l. you refer to note (G) page 293 in the appendix,

dix, and there give $\frac{1-a}{r}+\frac{1-b}{r^2}+\frac{1-c}{r^3}$ &c. $(t-1)$ $+\frac{1-P}{r^t}+\frac{1-P}{r^{t+1}}+\frac{1-P}{r^{t+2}}$ &c. for the exact value (I suppose) of the required assurance. This, if true, which however seems doubtful, is not sufficiently clear; for we are not told how far the latter series is to be continued; as it now stands it is indefinite; the former series, viz. $\frac{1-a}{r}+\frac{1-b}{r^2}$ &c. you have restricted to $t-1$ number of terms, and surely it was as necessary to limit the other, in order to its being properly understood. This value you farther say is equal to $\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$ &c. $(t)-\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}$ &c. $(t-1)+\frac{P}{r^t}+\frac{1-P}{r^t}\times\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$ &c. It is possible you may know it to be so; but is that sufficient? Should you not have shewn your readers also the truth of it? How are they to judge thereof, when you have left them entirely in the dark with respect to the latter part of the expression $\frac{1-P}{r^t}\times\overline{\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}}$ &c. I mean whether it is to be understood as continued in infinitum, or to some certain number of terms

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which you have omitted to declare? Were I to hazard a conjecture, I should apprehend the whole investigation in this note to be defective; nor can the rule for resolving the 14th question, given in page 48, possibly flow from the algebraical process to which you advert.—If I form a just Idea of the question you propose, it is to this effect.

A has 10l. per annum; this annuity, in case he lives 11 years beyond his present age (34 years) will then become certain to him and his heirs for ever. What sum in a present payment, by way of *assurance*, must A give to have this estate secured in perpetuity until the expiration of the said term?

*Solution*. Imagine C the insurer. It is evident that if A dies before the expiration of the proposed limit, viz. 11 years, in such a case C must, at the end of 11 years, pay A's heirs the full value of the estate, which we may here suppose 250l. or 25 years purchase, the rate being 4 *per cent*. and consequently the present worth of that sum, which is 162.4l. being multiplied by the probability that A shall decease before he attains the age

age of 45 years, viz. $\frac{103}{499}$ gives 33.52l. the price of the required aſſurance, which differs from the anſwer you give by 9l. 18s.

There cannot, I think, be any objection to this method of ſolution. It is extremely clear, that ſhould A deceaſe either at the end of the firſt, ſecond, third, &c. years within the propoſed limit, it can make no difference with reſpect to C's agreement, becauſe the heirs of A, had they been certain of the eſtate, could not poſſibly have got into poſſeſſion of it, until the limited time become expired; and therefore the inſurer is under no manner of neceſſity to pay the ſtipulated value of the eſtate, to the heir upon whom it may devolve, after the deceaſe of A, before the end of eleven years, from the time of agreement, ſhall be fully accompliſhed.

In the latter part of the example, to which the former you ſay is equivalent, it is required to find the preſent value of an annuity of 10l. for ever, to be entered upon at the failure of a life of 34 years, ſhould that happen within eleven years, or in other words (I ſuppoſe) it may be thus: A, who is now 34 years of age, has an eſtate of 10l. per annum; this eſtate,

ſhould

ſhould he happen to deceaſe before he attains to the age of 45 years, will go to B, and his heirs for ever, to find the value of B's expectation, rate of intereſt 4 *per cent*, and utmoſt extant of life being as before.

*Solution.* From the perpetuity 25, ſubſtract the value of a life of 45 years; multiply the remainder 14.2 by .6496, the preſent worth of 1l. to be received eleven years hence, and let this product, viz. 9.22432 be multiplied by $\frac{156}{299}$ the probability that a life of 34 ſhall reach to 45 years. To this product, which is 6.35, add 12.4, the value of the ſaid life of 34 years, the ſum 18.75 being ſubtracted from 25, the perpetuity leaves 6.25 years purchaſe, that is, 62l. 10s. for B's expectaion.

The whole of your ſolution to this 14th queſtion is therefore evidently wrong; nor indeed are the two caſes in the example the ſame; common reaſon points out to us, that the aſſurance for the perpetuity, to the end of eleven years, cannot be equal to the preſent value of the whole eſtate, to be entered upon at failure of ſuch a life, ſhould that happen in eleven years, but muſt certainly be leſs. Accordingly,

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we find it little more than half the latter value. But what ſeems very extraordinary, you have not here applied the neceſſary correction, ſo much inſiſted upon in the preceding queſtions, particularly in the 10th, where your anſwer differs from Simpſon's merely upon that account; for if that queſtion had paſſed uncorrected, it would have exactly agreed with his and Demoivre's, likewiſe with the ſolution in page 16 of this work, where, as well as in this example, no diſcount is allowed in caſe the propoſed life fails within the firſt year. Now how comes it to paſs that the correction is here omitted? Certainly if the 10th queſtion required correction, the example to the 14th, as its ſolution is upon the ſame ſuppoſition with regard to diſcount, ſhould, I think, ſtand in the ſame predicament. Be this as it may, I will venture to aſſert, the note G, in the appendix to which you refer, includes no demonſtration at all; nor is the rule you give for the method of ſolution a conſequence of the inveſtigation as has been already mentioned above; and, therefore, as you ſometimes apply your corrections, and ſometimes do not, your readers muſt be at a loſs to know when

they

they become neceſſary; for this very reaſon I conceive thoſe corrections might with great propriety have been intirely omitted.—Indeed the reaſons you aſſign for introducing of them ſeem rather of the whimſical kind; you make a ſuppoſition which can never, I think, take place in negotiating annuity buſineſs, rents of any and of every ſort are not paid until the tenants have been one year in poſſeſſion; and when you tell us that 25l. is the preſent value of an eſtate of 1l. per annum for ever, it is to be underſtood that the firſt rent is to be received a year hence; but if the buyer is to receive 1l. at the time of making the contract, it is then equivalent to 26l. Is not this telling your readers that the value of an eſtate, together with 1l. is worth juſt 1l. more than it would have been without it?—which is certainly as true as any propoſition in Euclid, and I believe more univerſally known, conſequently leſs eſſential to Dr. Price's treatiſe on *Reverſionary Payments*, which it may be ſuppoſed was intentionally written with a view to inſtruct us in the more difficult parts of this uſeſul branch of mathematical knowledge, rather than to amuſe the reader with needleſs obſer-

vations

vations upon the moſt evident and ſatisfactory axioms in nature. Beſides, coming from Dr. Price, they receive additional conſequence, and his readers will even diſtruſt their own common ſenſe, when they find the learned Dr. himſelf judged it neceſſary to expatiate upon principles hitherto ſo unanimouſly aſſented to.

The third eſſay, page 228, which relates to the method of calculating the values of reverſions depending on ſurvivorſhip, deſerves particular attention, it being a matter of much importance; and, as you juſtly obſerve, any conſiderable errors in the method of ſolving ſuch queſtions, muſt in time produce very bad conſequences; for this reaſon you have here directed your views towards diſcovering ſuch miſtakes as may poſſibly have eſcaped the notice of all other former writers upon the ſame ſubject; and, in conſequence of this enquiry, have pointed out a particular (ſuppoſed) error into which there is danger of falling, in finding the value of ſuch reverſions as depend upon ſurvivorſhip. You ſeem farther to be of opinion, that Mr. Demoivre has moſt eſſentially failed in his computations relating to this particular branch of the doctrine of *annuities upon*

*lives*. In order to juſtify the opinion you have formed of that late celebrated author, you adduce a problem, which, I ſuppoſe, is ſimilar to the 17th and 20th in ſome former addition of Demoivre's treatiſe, as the 17th and 20th problems in the fourth addition are quite of a different nature. However, I make no ſort of doubt concerning the truth of your aſſertion, and am ready to believe the queſtion ſtill remains unaltered. You do not indeed ſay it is actually to be found in his treatiſe, but only remark, that, in order to be as plain as poſſible, the following caſe is neceſſarily propoſed: " A, aged 40, expects to come to the poſſeſſion of an eſtate, ſhould he ſurvive B, aged likewiſe 40." In theſe circumſtances he offers, in order to raiſe a preſent ſum, to give ſecurity for 40l. per annum out of the eſtate at his death, provided he ſhould get into poſſeſſion; that is, provided he ſhould ſurvive B. What is the ſum that ought now to be advanced to him in conſideration of ſuch ſecurity, reckoning compound intereſt at 4 *per cent.*? The ſolution to this problem, as it depends upon Mr. Demoivre's directions in his treatiſe on annuities, problem 17th and 20th, you ſay, lead us to ſeek the

the preſent ſum A ſhould receive for the reverſion of 40l. *per annum* for ever after his death, ſuppoſing it not dependent on his ſurviving B. But the lender having a chance to loſe his money, a compenſation ought to be made to him for the riſk he runs, which is founded on the poſſibility, that a man of 40 years of age may not ſurvive another perſon of the ſame age. This chance is an equal chance, and therefore half the ſum which A ſhould have received in the former circumſtance, unreſtricted by the ſurvivorſhip, is the juſt value of the required ſum, and in the caſe propoſed equal to 236l. This ſolution you acknowledge has a plauſible appearance, and ſeem of opinion, moſt perſons will probably be ready to pronounce it right; nor will this, you ſay, be at all wonderful, as ſo great a maſter of theſe ſubjects as Mr. Demoivre appears to have been miſled by it; to which you add—Nothing is more neceſſary to prove it to be fallacious, than proceeding in the ſame way to ſolve the following ſimiliar queſtion. A, aged 40 years, offers to give ſecurity for 40l. per *annum* to be entered upon at his death, provided it ſhould happen before the death of B, aged

likewiſe 40. What ſum ſhould be advanced to him for ſuch a reverſion intereſt being reckoned at 4 *per cent.*?

The anſwer, agreeable to the method juſt deſcribed, comes out the very ſame with that already given. This I think is not at all to be wondered at, as there does not ſeem to be any ſort of difference whatſoever between this and the former queſtion. And yet you tell us, it may be eaſily ſeen that this muſt be wrong "for the value of a reverſion to be received, when a perſon of a given age dies, cannot be the ſame, whether the condition of obtaining it is that he ſhall die *before* or that he ſhall die *after* another perſon." Why not, if the ages are equal? I grant, that a perſon, who is certain of coming into poſſeſſion of an eſtate, would rather have it ſooner than later; but ſtill the expectation thereon may be the ſame, and the value in either caſe equivalent. To put this matter beyond the poſſibility of a doubt, let us ſuppoſe, that C is to poſſeſs, in perpetuity, an eſtate of a certain value in caſe A, who is of a given age, ſurvives B of equal age with himſelf. And again, imagine the ſame perſon C is intitled

intitled to the perpetuity of another eſtate equal to the former, in caſe B ſurvives A. To which do you think the preference is due? You might perhaps take the former, becauſe you apprehend the reverſion in the latter caſe, muſt, without doubt, be of leſs value than in the former. The reaſon you aſſign for this conjecture is, that the purchaſer may ſooner get into poſſeſſion. There certainly cannot be any juſt foundation for this opinion; becauſe, as the lives are equal, can it be ſuppoſed, that more time will be required to produce one ſurvivorſhip rather than the other? that is, can it be expected the purchaſer C will in leſs time come into poſſeſſion of the eſtate upon condition that A ſurvives B, than if the agreement had provided B for the ſurvivorſhip inſtead of A? As you ſeem to lay the whole ſtreſs of your objections to Mr. Demoivre's ſolutions of theſe queſtions, upon the ſingle circumſtance above mentioned, I am in ſome meaſure inclined to think your obſervations in this eſſay are intirely defective; and likewiſe, that you have juſtified my conjectures, by endeavouring to eſtabliſh your own in the following manner.

Demoivre's

Demoivre's firſt queſtion reſolves itſelf (you ſay) into the following general queſtion, viz.

What is the preſent value of a given reverſionary eſtate to be entered upon after the failure of two lives, provided *one in particular* of them ſhould be the *longeſt life?*

In the ſolution to this queſtion you find the preſent value of an eſtate to be enjoyed for ever, after the failure of the longeſt of the two lives, and multiply it by the annual rent, the product you ſay, would be the anſwer, were the eſtate certainly to be enjoyed after the extinction of the longeſt of two lives, both 40, that is, whether one or the other of them failed firſt, and half this certainty, becauſe it is an even chance that A's life in particular ſhould fail laſt, rather than B's, is the true value of the reverſion. The preſent value of an eſtate to be enjoyed for ever, after the failure of the longeſt of the two equal lives, is a circumſtance not mentioned in the queſtion; nay, ſuch a ſuppoſition would render the queſtion impoſſible; for as one of the lives, ſuppoſe A's is in particular to be the longeſt, otherwiſe the expectation upon the

the eſtate is totally loſt to the heirs in perpetuity. It ſeems abſurd to introduce the ſurvivorſhip of B into the ſolution, as it makes entirely againſt the eſtate ever falling to the purchaſer, or his heirs, after the deceaſe of A. The queſtion, to which your anſwer ſeems adapted, is of this kind.

Two perſons, E and F, have equal expectations for themſelves and heirs to enjoy an eſtate of 40l. *per annum* (continued for ever) to be entered upon after the extinction of the longeſt of two equal lives, both 40 years of age; required the preſent value of their expectations.

Your next queſtion is this: " What is the preſent value of 40l. *per annum* for ever, to be entered upon after the extinction of two joint lives, both 40, that is, whenever *either* of them ſhall fail, provided the firſt that fails ſhould happen to be A's life in particular? The anſwer is found (you ſay) by ſubtracting the preſent value of the *two joint* lives from the *perpetuity*, and multiplying by $\frac{1}{2}$, or by the chance that A in particular ſhall die firſt. And this will give the required value 303.4l.

With

With regard to this queſtion there can be no ſort of doubt as to the meaning of it; you have clearly ſet it forth properly defined. An annuity of 40l. is to be entered upon immediately upon the deceaſe of either A or B, no matter which ſhall happen to die firſt, and therefore the purchaſer or his heirs are ſure of having the eſtate, becauſe it is a certainty, that A or B will die firſt, unleſs they both deceaſe at the ſame moment of time, which is almoſt impoſſible, the probability being ſo extremely ſmall, that it may very reaſonably be excluded the computation, and therefore the purchaſer, as yet may be ſaid to have by agreement bought a certainty. Thus it ſtands, until you introduce the contradictory clauſe, namely, that unleſs the joint exiſtence is terminated by the deceaſe of A, the purchaſer muſt wave all manner of claim to the reverſion. I would here aſk, how is this queſtion propoſed in a conſiſtent manner, when the firſt condition therein intitles the purchaſer, or his heirs, to a certain eſtate upon the failing of either of two aſſigned lives, which muſt inevitably happen, and yet at the ſame time he is

told

told, that unleſs A ſurvives B, which may not happen, he muſt give up all pretenſion to it. Surely this ſeems to imply a particular circumſtance is to be, and not to be, at one and the ſame time.

Here in order to render the queſtion poſſible, and the anſwer you have given to it true, it ſhould have been thus expreſſed. Two perſons, E and F, have equal expectation for themſelves and heirs to enjoy an eſtate of 40l. *per annum* (continued in perpetuity) to be entered upon after the extinction of two *joint lives*, both 40, that is, whenever either of them ſhall happen to fail; required the preſent value of E or F's expectation.

Believing yourſelf right, you have adduced ſome kind of proof in ſupport of the arguments uſed in this eſſay to prove Demoivre's hypotheſis defective. This attempt has, I doubt, led you into farther error, whereby you extirpate a true ſolution to a very poſſible queſtion, and confirm the ſolution to an impoſſible one, eſpecially when you ſay, "In ſhort it appears in both caſes, that, according to the firſt method of ſolution, we are to ſubtract from the per-

petuity, the value of one of the single lives; when, in the former case, the value of the longest of the two lives, and, in the latter case, the value of their *joint* continuance ought in reality to be subtracted." To this you add, "I need not say what prodigious errors may often arise from hence, and how unfit such a method (Demoivre's) of solution is for practice." There seems something very ænigmatical in this conclusion, namely, that prodigious errors should arise from true principles of solution, and immediately subside upon the application of other principles which are not true. I doubt not but you would, in answer to these animadversions upon your essay, introduce Mr. Simpson as coinciding with yourself in opinion concerning these reversionary purchases, from what he says in his select exercises; where, at page 322, he speaks on this subject in the following manner. "I have been very particular on these kinds of problems; and the more so, as there has been no method before published, that I know of, by which they can be rightly determined; 'tis true, the manner of proceeding by first finding the probability of survivorship,

(which

(which method is uſed in my former work, and which a celebrated author has largely inſiſted on in three ſucceſſive editions) may be applied to good advantage, when the given ages are nearly equal; but then it is certain, that this is not a genuine way of going to work, and that the concluſion hence derived are at beſt but near approximations."

This excellent mathematician has here (you think) expreſſed himſelf much too favourably of the method of ſolution on which you have remarked: granting this to be true, it makes nothing for your purpoſe: Mr. Simpſon has no where denied the truth of Mr. Demoivre's ſolution of the queſtion to which you advert; indeed he could not with propriety, it being moſt indiſputably a true one; and how far Simpſon's declaration, to which you have appealed, will ſerve to juſtify propoſing an impoſſibility, ſuch I think is the latter of thoſe caſes which you have cited in the eſſay, ſeems not very difficult to conjecture. Let us now take a nearer view of this matter, in order to diſcover whether any thing adverted to by that great mathematician in his exerciſes can poſſibly

make for your ſide of the argument.—The moſt probable of this kind I take to be his 33d problem, and ſeems directly a caſe in point, as the lawyers have it. Here C and his heirs are intitled to an eſtate of a given value, upon the deceaſe of B, provided B ſurvives A; to find the value of this expectation in preſent money. The ſolution which Simpſon gives to this queſtion he intended but as an approximation, it being marked as ſuch, and therefore the value of an annuity upon the longeſt of two equal lives, entering into the computation, was merely adventitious; for had the ſolution been confined to exactneſs, it would then have been definite, and have appeared in another form: you were I believe deceived by it, the queſtion being ſo very like to Demoivre's (though not perfectly the ſame) and the direction to find the value of an annuity upon the longeſt of two *equal lives* ſtanding in the front of the ſolution, might induce you too haſtily to conclude, that no ſolution whatever, to Mr. Demoivre's queſtion, that did not begin with computing the value upon the *longeſt* of two equal lives, could poſſibly be right; and therefore as there remained

mained with you no ſort of doubt of his being miſtaken, you immediately ſuggeſted a correction neceſſary.

It is, I believe, owing, in a great meaſure, to inattention with regard to the true meaning or import of Simpſon's inveſtigations, that you ſo frequently imagine they ſtand in need of ſome kind of correction; otherwiſe, how could you poſſibly be ſo miſtaken as to not only tell your readers that the anſwer to the 12th queſtion, page 39, in your work, which relates to an inſtitution for the relief of widows, is deduced from the 33d problem in his Select Exerciſes, but alſo, that the problem and its ſolution are given by Mr. Simpſon in the following words.-- "A *and his heirs are entitled to an eſtate of a given value upon the deceaſe of* B, *provided* B *ſurvives* A; *to find the value of their expectation in preſent money*." There is certainly an impropriety in the propoſal; for how is it poſſible that A can be intitled to an eſtate after he is dead? It is true, his heirs, as ſucceſſors, may; but to ſay that A and his heirs are intitled to the eſtate after the deceaſe of B, provided A dies firſt, is ridiculous. However, I am very ſure there

there is no ſuch problem in his exerciſes; notwithſtanding you tell us, the ſolution is thus: "Find the value of an annuity on the longeſt of two equal lives, whereof the common age is that of the older of the lives A and B; which value ſubtract from the perpetuity, and take half the remainder, then it will be as the expectation of duration of the younger of the lives A and B, is to that of the older, ſo is the ſaid half remainder to the number of years purchaſe required, when the life B *is the older of the two*. But if B *be the younger*, then, to the number thus found, add the value of an annuity on the longeſt of the lives A and B, and ſubtract the ſum from the perpetuity, for the anſwer in this caſe."

If the eſtate is 4l. *per annum*, the age of B 40, and of A 32, intereſt 4 *per cent*. the anſwer by this rule comes out l.14.35, which divided by 10.4 (your preſcribed correction) gives l.13.8, the value in anſwer to the 12th queſtion, or the ſum to be paid by a widower; ſo that at his deceaſe his family ſhall be entitled to 100l. But if B is 30, and A 40, the ſame value is 20l.

The

The method of ſolution you have here applied to this queſtion is the very ſame with the directions given by Simpſon at page 299 of his exerciſes to a problem of a very different kind to that above mentioned, and in which there is no ſort of impropriety involved; it is clearly expreſſed in this manner: C *and his heirs are intitled to the poſſeſſion of an eſtate of a given value upon the deceaſe of* B, *provided* B *ſurvives* A, *to find the worth of their expectation in preſent money*. And to this problem Mr. Simpſon's ſolution, which you, by ſome ſort of miſtake, have placed for the anſwer to your 12th queſtion, is an elegant approximation, and will remain ſuch, if permitted to paſs without your uſual correction, ſo often inſiſted upon in the courſe of your performance.

There appears ſomething very extraordinary towards the concluſion of this third eſſay; where, after having told us, that Mr. Simpſon has expreſſed himſelf much too favourably of the method of ſolution, on which you have remarked, you immediately add, that he is himſelf miſtaken, and his " obſervations in this paſſage are only true when applied to a different method

method uſed in the 28th and following problems of his treatiſe on the doctrine of *Annuities* and *Reverſions*, and even this method is exact only when the lives are equal; but it gives reſults which are too far from the truth, when there is any conſiderable inequality between the lives."

It is (you ſay) with reluctance you have made ſome of theſe remarks. Mr. Demoivre has made very important improvements in this branch of ſcience; and the higheſt reſpect is due to his name and authority. This, however, only renders theſe remarks more neceſſary." As this part of the concluſion to the eſſay comes with an air of conſiderable importance, and indicates no leſs than having therein pointed out, refuted, and corrected Demoivre's miſtakes in the ſolution of various caſes relating to the values of reverſions, it may be, perhaps, worth while to examine the methods by which you have effected ſo deſirable an acquiſition. Firſt then: In the beginning of this eſſay, you have propoſed a queſtion relating to ſurvivorſhip, of a ſimilar kind with one of Mr. Demoivre's, and have given a very juſt anſwer

to

to it, agreeable to his directions for that purpose. Secondly, You have changed the form of the question to another of the same kind, retained the sense of the former, and have given also a just solution to it as before. Thirdly, You have by a most unjustifiable transition converted the former problem to what you call a general case (page 232) which seems to bear no sort of affinity with Demoivre's, and have given a solution to it, which, instead of containing the proper answer, appertains to a question of a very different nature. Fourthly, you have very ingeniously resolved Mr. Demoivre's question into another which includes an impossibility, viz. a particular circumstance, to be, and not to be, at the same time. And lastly you have told your readers, that, to have a true solution to the cases proposed by Demoivre (one of them, the latter, seems rather of your own making) The value of the *longest* of the two lives in the former, and their *joint* continuance in the latter case, ought to be subtracted from the perpetuity, and the remainder multiplied by $\frac{1}{2}$, to produce the true answer.—You finally conclude with observing, " That in the first chapter (questions 10th,

11th, 12th, 14th. &c.) is given a very minute account of the method of finding, in all cases, the values of reversions which have been the subject of this essay."—The solution to the 11th, question would have been right, had you omitted the division by 1l. increased by its interest for a year. The other questions in general are answered in a manner very defective, if not totally wrong.

I readily agree with what you say concerning the 28th, 29th, and some other problems in Simpson's treatise on the *doctrine of annuities* and *reversions*, namely, that his method is exact only when the lives are equal; but how doth this agree with what immediately preceeds? to wit, in both cases you have specified the ages are equal, and yet you tell us in one of them the error is a good deal above a third of the true value, and in the other a fifth. I must confess this appears very extraordinary, and really well worth inquiring into, in order to find out where the error can possibly be. Mr. Demoivre's answer is 236l. In the former case of your answer to his question we have 168.6l. and in the

the latter, the ſolution brings out 303.4l. for the anſwer. I do not indeed know which of theſe three anſwers you deſign for the true one: however, the firſt 236l. I imagine to be quite out of the queſtion, and therefore it muſt either be 168.6, or 303.4; if the latter, then Demoivre's ſolution differs from the truth by 67.4l. the error being in defect, and is more than one third of itſelf leſs than it ſhould be: If on the other hand we are to take 168.6l. for the criterion, his anſwer becomes erroneous in exceſs juſt as much, viz. 67.4l. as it was before in defect. Now if Mr. Simpſon's method, which you ſeem to have followed, be exact when the the lives are equal, how comes it about, that Demoivre's, whoſe enquires concerning this ſubject extend no farther than the caſes in which equal lives are concerned, ſhould happen to be wrong alſo? One would really be led to imagine, by the account you give of this matter, that Simpſon cannot be quite right, even upon the ſuppoſition of equal lives; however, be this as it may, I think it will not be very difficult to clear Demoivre from the imputation of having made any miſtake in the ſolution of the queſtion in debate, by an algebraical pro-

ceſs, and which I ſhall endeavour to do, after having made ſome neceſſary obſervations upon the note (N) in the appendix, to which the reader is referred, for a complete demonſtration of theſe ſolutions.

In this note (N) you ſay, it is plain, that the purchaſer of A's right, as ſtated in the firſt of the queſtions to which this note alludes, cannot get into poſſeſſion till the year when A and B ſhall be both dead, (there ſeems to be no reaſon for this limitation, viz. "both lives failing in the ſame year") nor then, unleſs A happens to die laſt. Now ſuppoſing the common complement of life to be $n$, the probability that A and B ſhall be both dead at the end of the firſt year, and A die laſt, is $\overline{1-\frac{n-1}{n}} \times \overline{1-\frac{n-1}{n}} \times \frac{1}{2} = \frac{1}{2} - \frac{n-1}{2n} - \frac{n-1}{2n} + \frac{\frac{\overline{n-1}^{2}}{2}}{2n}$ This I agree to, as the lives are equal. — In like manner the probability that they ſhall be both dead at the end of the ſecond, third, &c. years, and A die laſt, is $\frac{1}{2} - \frac{n-2}{2n} - \frac{n-2}{2} + \frac{\frac{\overline{n-2}^{2}}{2}}{2n}$ $\frac{1}{2} - \frac{n-3}{2n} - \frac{n-3}{2n} + \frac{\frac{\overline{n-3}^{2}}{2}}{2n}$ &c. Theſe expreſſions, I think, are not true, becauſe each ſucceſſive probability includes

includes the preceding; and, therefore, notstanding $1-\frac{n-2}{n}-\frac{n-2}{n}+\frac{\overline{n-2}^{2}}{n^{2}}$ or its equal $\frac{4}{n^{2}}$ is most certainly the probability of both lives failing within the expiration of two years; yet it doth not restrict that event to the last of these years, which should have been done; otherwise, in summing the terms, each foregoing probability is twice taken into the computation, which, I think, renders the whole process defective. There yet seems to be another objection to your conclusion; I mean, to that part of it, where you mention "the sum of the terms continued in infinitum;" for, if this is to be the case, the last of the numerals, 1, 2, 3, &c. will, as they advance, soon exceed the complement of life; the more remote terms change from positive to negative, or *vice versa*, and the whole order of the series thereby become reverted. It is true, that $\frac{1}{2r}+\frac{1}{2r^{2}}+\frac{1}{2r^{3}}$ &c. in infinitum, is equal to half the perpetuity; but

$$\frac{n-1}{2nr}+\frac{n-1}{2nr}-\frac{\overline{n-1}^{2}}{2n^{2}r}+\frac{n-2}{2nr}+\frac{n-2}{2nr^{2}}-\frac{\overline{n-2}^{2}}{2n^{2}r^{2}}+\frac{n-3}{2nr^{3}}+\frac{n-3}{2nr^{3}}-\frac{\overline{n-3}^{2}}{2n^{2}r^{3}}$$

&c. which is half the value of the joint lives subtracted from half the sum of the values of two

two ſingle lives, that is, half the value of the longeſt of the two lives muſt terminate in a certain finite number of terms, viz. $n-1$; and therefore, to take half the value of the longeſt of the two lives from half the perpetuity, in order to obtain an anſwer to the queſtion as propoſed, ſeems arbitary, for your inveſtigation certainly includes no ſuch direction. The theorem for the preſent value of the eſtate, or ſum of the reverſionary rents, is by your own proceſs $\frac{1}{2r}+\frac{1}{2r^{2}}+\frac{1}{2r^{3}}$ &c. *minus* half the value of the *longeſt* of the two lives, where the numb. of terms of the progreſſion $\frac{1}{2r}+\frac{1}{2r^{2}}+\frac{1}{2r^{3}}$ &c. muſt evidently be the ſame with the number of terms in either of the ſeries which conſtitute the ſaid value of the longeſt life; and, therefore, your introducing half the perpetuity is without any authority for ſo doing. This, I apprehend, plainly ſhews, that all you have advanced in the note (N) in ſupport of your ſolutions to Demoivre's queſtion, is entirely void of any foundation in the true principles of mathematics. But leaſt any doubt ſhould remain with reſpect to this aſſertion, let us now reſume the proceſs, by taking the

the probabilities as they ariſe in their moſt contracted ſtate; this will make no difference in the reſult, eſpecially as the *form* of the rule, you mean here to demonſtrate, is not our immediate enquiry: This allowed, we have $\frac{1}{2nr^{2}} + \frac{4}{2nr^{2}\,{}^{2}} + \frac{9}{2nr^{2}\,{}^{3}} + \frac{16}{2nr^{2}\,{}^{4}}$ &c. continued to $n-1$ number of terms; and this (were your hypotheſis true) would be the value required, and is univerſally equal to $\frac{1}{2n \times \overline{r-1}^{\,2}}$ into $\dfrac{2r^{2} - \frac{2}{r^{p-2}}}{\overline{r-1}^{\,2}} - \frac{2p}{r^{p-1} \times \overline{r-1}^{\,2}} - \frac{p^{2}}{r^{p}} - \dfrac{r - \frac{1}{r^{p-1}}}{r-1}$ where $p$ repreſents the number of terms to be taken of the propoſed ſeries in this caſe equal to $n-1$.

This theorem, which is certainly the true reſult ariſing from your hypotheſis, differs widely from the determination which you give as a ſolution of the queſtion, viz. *half the perpetuity leſſened by the value of the longeſt of the two equal lives.* In ſhort, this note, I am of opinion, contains no ſort of inveſtigation whatſoever eſſential either to the ſolution of Demoivre's queſtion, or the general propoſition into which, you tell your readers, it may naturally be

be resolved.—It is true, your solutions to the questions, if proposed in the manner I have mentioned, stand perfectly right; but even then those answers are not deducible from any thing which appears in this note, to which you advert; and therefore I cannot help thinking you are here intirely mistaken throughout. However, I shall pursue the enquiry no farther, but conclude these animadversions, upon the third essay, with reproposing Mr. Demoivre's question, and giving a just solution of it.

The question as before; A, aged 40, expects to come to the possession of an estate, should he survive B, aged likewise 40. In these circumstances he offers, in order to raise a present sum, to give security for 40l. per *annum* out of the estate at his death, provided he should get into possession, that is, provided he should survive B. What is the sum that ought now to be advanced to him in consideration of such security, reckoning compound interest at 4 *per cent*?

*Solution*. It is evident, that an annuity, to continue in perpetuity, whenever it shall be

entered

entered upon, is always of the ſame value; that is, if I have an eſtate to me and my heirs for ever, I can ſell it for the ſame ſum, whether I diſpoſe of it immediately, or any number of years hence, and conſequently a freehold of 40l. *per annum* will, allowing 4 *per cent* to be worth 25 years purchaſe, or 1000l. whenever it ſhall be offered to ſale. This being admitted, let C become the purchaſer for himſelf and his heirs in perpetuity, of the aſſigned 40l. *per annum*, or which is the ſame thing its equal value 1000l. in a ſingle ſum, to be received at the death of A, if he has ſurvived B. Now by this purchaſe being made, it is evident, that C, or his ſucceſſor, becomes intitled to the receipt of 1000l. upon the deceaſe of A, provided B's life be then extinct. But on the other hand, the expectation of C and his heirs, to poſſeſs the eſtate at the deceaſe of A, entirely ſubſides; if B ſhould be then living. Let *r* and *n* repreſent as before, the rate of intereſt and common complement of their equal ages, put 1000l. = S. If A happen to die within the firſt year from the contract, of which the

H probability

probability is $\frac{1}{n}$. C the purchaser becomes intitled to the sum S, provided B be also then deceased, otherwise C loses his whole purchase money. Now it being an equal chance whether B survives A, or A survives B, it is plain that $\frac{1}{2n} \times S$ is the exact value of C's expectation upon the contingent circumstance of receiving the sum S within the first year after the purchase. In like manner the value of his expectation upon the said sum S to be received at the expiration of two years, will be $\frac{1}{2nr} \times S$, at end of three and four years $\frac{1}{2nr^2} \times S$ and $\frac{1}{2nr^3} \times S$ respectively; from hence the law of continuation becomes manifest; and, as the number of terms in the series must evidently be equal to $n$, the sum thereof will, by a well known theorem, be expounded by $\frac{r - \frac{1}{r^{n-1}}}{r-1} \times \frac{S}{n}$ which is the total value of C's expectation. But as $\frac{1}{r-1} - \frac{r - \frac{1}{r^{n-1}}}{\overline{r-1}^2 \times n}$ (where $\frac{1}{r-1}$ represents the perpetuity) is well known to denote the present worth of an annuity of 1l. to continue during a life, whose complement of age is $n$ years; it is clear, that the

the value of the propoſed life, being ſubtracted from the perpetuity, and the remainder, which is here $\dfrac{\dfrac{r-\frac{1}{n-1}}{r}}{\dfrac{2}{r-1}\times n}$ multiplied by $\frac{r-1}{2}\times S$, gives the required anſwer, and is equivalent to Demoivre's, deduced from his rule for the ſame purpoſe, viz. *From the perpetuity ſubtract the value of the life, and multiply the remainder by the annual rent, and divide the product by 2, the quotient gives the value required.*

At page 147 you make ſome remarks upon a method which has been propoſed, for diſcharging the national debt by life annuities, and in order to ſhew the inſufficency of this expedient, you adduce the following obſervations.

"Let us ſuppoſe, that 33,333,000l. is to be paid off, by offering to the public creditors life annuities, in lieu of their 3 *per cents*. A life of 60, as in the *Breſlaw*, *Norwich*, and *Northampton* tables, of obſervation is worth 11 years purchaſe. A life at 30, is worth 16½ years purchaſe, certainly therefore, no ſcheme of this kind would be ſufficently inviting, which did

 not

not offer 8 *per cent* at an average to all ſubſcribers. Let us however, ſuppoſe, that no more than $7\frac{1}{2}$ is given; and that there are 33333 ſubſcribers, at 1000l. ſtock each, for which a life annuity is to be granted of 75l. or for the whole ſtock ſubſcribed two millions and a half. A million and a half extraordinary, therefore muſt be provided towards paying theſe annuities. Let us farther ſuppoſe that the ſubſcribers are perſons between the ages of 30 and 60: and that the numbers of them, at all the intermediate ages, are in the ſame proportions to one another, with the proportions of the living at theſe ages, as they exiſt in the world, or, as they are given in *tables* of *obſervation*. Let us again ſuppoſe, that as theſe annuitants die off, they are immediately replaced by others, who are continually offering themſelves at the ſame ages, and in the ſame proportional numbers at theſe ages, with thoſe of the original ſubſcribers at the time they ſubcribed; in conſequence of which, the whole number of annuitants will be kept always the ſame. In theſe circumſtances, it will be 30 years, at leaſt, before a number will die off equal to the whole

number

number, that is, before 33 millions of debt will be annihilated.

In a reference at the bottom of page 149, you ſay " a demonſtration of this will be given in the appendix, note (K)." In conſequence of this declaration, we might naturally expect to find in that note, ſome kind of proof of what is here advanced, concerning the term of 30 years being the requiſite time for annihilating the propoſed part of the public debt. Let us now ſee what it contains, and how far it may be ſaid to include the demonſtration in conſequence of the promiſe made at page 149, in the body of your work?

The note (K) appendix, p. 297, runs thus:

Let $d$ ſignify the difference between the *complements* of the youngeſt and the oldeſt life in the body of annuitants, here deſcribed, at the time they enter; let S ſignify the ſum of theſe *complements*, $n$ any given number of years not not greater than $\frac{S}{2}-\frac{d}{2}$; and $x$ the ratio of the whole number of annuitants to $\frac{S \times d}{2}$. Then

$x \times d$

$x \times d$ will be the number that will die the 1ſt year.

$x \times \overline{d + \frac{2d}{S}}$ the number that will die the 2d year.

$x \times \overline{d + \frac{4d}{S} + \frac{4d}{S^2}}$ 3d year.

$x \times \overline{d + \frac{6d}{S} + \frac{8d}{S^2} + \frac{8d}{S^3}}$ 4th year.

$x \times \overline{d + \frac{8d}{S} + \frac{12d}{S^2} + \frac{16d}{S^3} + \frac{16d}{S^4}}$ 5th year. And

$$x \times nd + \overline{n^2 - n} \times \frac{d}{S} + \overline{\overline{n-2} \times \overline{n-2}^2} \times \frac{2d}{S^2} + \overline{\overline{n-3} + \overline{n-3}^2} \times \frac{4d}{S^3} \text{ \&c.}$$

(n) will be the whole number dying in $n$ years. When $n$ is greater than $\frac{S}{2} - \frac{d}{2}$, this ſeries is greater than the whole number dying in $n$ years; but in all other caſes it gives this number exactly, ſuppoſing the probabilities of life to decreaſe uniformly.—In the preſent inſtance, the youngeſt life being 30, and the oldeſt 60 years, the two complements are 56 and 26. $S = 82$, $d = 30$. $\frac{S \times d}{2} = 1230$. And therefore $x = \frac{33.333}{1230} = 27.1$. Take $n = 30$ years, and the foregoing ſeries will be ———

$$27.1 \times \overline{900 + 318.2 + 7.242 + 1.64} = 33{,}214,$$

which is a little greater than the whole number dying in 30 years, but at the ſame time leſs than the whole number of annuitants.

It appears rather extraordinary that you ſhould tell your readers, this note (K) contains

a de-

a demonſtration of what is advanced at page 149, concerning the time, viz. 30 years, in which it may be expected that a number of annuitants, neceſſary to the diſcharge of 33 millions of the public debt, will die off, as the proceſs in this note, doth not bear even the leaſt reſemblance to a demonſtration of any kind, it can at moſt be only ſaid to exhibit a theorem for finding the annual number of deaths in the whole body of annuitants, and if you intended it as a true and general theorem for that purpoſe, I think you have not ſucceeded, becauſe there are caſes wherein it will prove defective, nay indeed, if we take that particular caſe in which the lives are all equal, be the common age what it will, the yearly deaths do entirely vaniſh, and conſequently your 33333 annuitants, are by this theorem rendered immortal; which is I think, a ſcheme ſufficiently inviting, notwithſtanding your opinion, at page 148, that no ſcheme would be ſo, which did not offer 8 *per cent* at an average to all ſubſcribers. However, as I cannot ſuppoſe you are in earneſt with regard to the application of your theorem to all caſes indeſcriminately,

indescriminately, or indeed to the case I have adverted to in particular. I shall take the liberty to point out (from Demoivre's principles,) a method which will serve the purpose very nearly in all cases whatsoever, which is this.

Take a mean of all the proposed lives of the annuitants, that is divide the sum of their respective ages by the number of the subscribers, let this be considered as a single life, and take its complement to the extremity of old age. Multiply the complement by the number of deaths which are to happen, and divide the product by the whole number of annuitants increased by unity, the quotien will give the time required. To apply this rule to your inquiry concerning the time for annihilating the proposed sum of 33,333,000l. as part of the national debt by granting life annuities to 33.333 subcribers of 1000l. stock each as before, where the oldest life is 60, and the youngest 30, we shall have 45 years for the mean age nearly, which taken from 86, the supposed utmost extent of human life, leaves 41 for the complement, this multiplied by 33.333 and the product divided by 33.334, gives 40.9987, the

number

number of years required, which exceeds your determination of the required number by almost eleven years.

In note (L) page 301, you define the expectation of two unequal joint lives by $\frac{m}{2}-\frac{m^2}{6n}$, where *m* is the complement of the oldest life, and *n* the complement of the youngest. By this enquiry, extended to the expectation of survivorship, you determine the whole expectation thereof to be $\frac{n}{2}-\frac{m}{2}+\frac{m^2}{3n}$ The expectation of survivorship on the part of the oldest $\frac{m^2}{6n}$ and on the part of the youngest $\frac{n}{2}-\frac{m}{2}+\frac{m^2}{6m}$ As you do not give the investigation of these conclusions, but only remark in general terms, that they are deducible from the same fluxional calculus, as is made use of in the former part of this note. I have examined very particularly into the matter, and find, that the first expression for the joint lives, viz. $\frac{m}{2}-\frac{m^2}{6n}$ is most undoubtedly right, and the other $\frac{m^2}{6n}$, $\frac{n}{2}-\frac{m}{2}+\frac{m^2}{6n}$ which denote the respective expectations of survivorship on the part of the oldest and youngest life, are undoubtedly wrong, if we may believe

either Simpſon or Demoivre, who both prove, that the ſum of the expectations, which each has for ſurvivorſhip, do conſtantly make unity; whereas you have ſubtracted the expectation of the joint exiſtence from the expectation of each ſingle life for the reſpective expectations required; and, it is very clear, the ſum of thoſe remainders doth not conſtitute the certainty, or, which is the ſame thing, is not equal to unity. Notwithſtanding your inveſtigations are ſo very defective, yet you apply theſe concluſions to the moſt important articles in your performance; nay, you even illuſtrate the error by an example towards the end of the note, by enquiring the number of marriages in being together that will, in $x$ years, grow out of one yearly marriage between perſons of *unequal* ages, the complement of the oldeſt life being $m$, and of the youngeſt $n$. This number, you ſay, will be $\frac{x^3}{3nm} - \frac{\overline{n+m} \times x^2}{2nm} + x$, and if the number of years is required, in which any given number of yearly marriages between men and women at given ages will increaſe, ſo far as to be in any given proportion to the greateſt number that can poſ-

ſibly

ſibly grow out of ſuch marriages ; this expreſſion muſt be made equal to the *expectation* of their joint lives, or of each marriage, multiplied by the fraction expreſſing the given proportion, and the root of the equation will be the anſwer. Thus it may be found, that one marriage every year, between perſons 33 and 25 years of age, would, in 10 years, increaſe to 8.35, in 15 years to 11.38, and in 53 years to 19, or their greateſt poſſible number ; and, conſequently, that 35 ſuch yearly marriages would, in 10 years, increaſe to 292, in 15 years to 398, and in 53 years to 665.—And if it is inquired in what number of years 35 ſuch yearly marriages would increaſe to half the number in being together, poſſible to be derived from them, the value of *x*, in the cubic equation $\frac{x^3}{3nm} - \frac{\overline{1+n} \times x^2}{2nm} + x = \frac{m}{2} - \frac{m}{6n} \times \frac{1}{2}$ muſt be found, which, in the preſent inſtance, is nearly 12.

The whole of this operation I take to be entirely wrong, or, which is the ſame thing, it has, in the ſenſe you apply it, no ſort of meaning at all. You will, perhaps, contend, that $\frac{x^3}{3nm} - \frac{\overline{n+m} \times x^2}{2nm} + x$, is a true expreſſion, and fairly

deduced from undeniable principles by a fluxional investigation. It is so, I grant; but yet, unless $x$ becomes the complement of the oldest life, it determines nothing, when $x = m$ the expression becomes $\frac{1}{2}m - \frac{m^2}{6n}$ the number of years the joint existence may be expected to continue, and is all that can be hence determined. To say it expounds the number of marriages that will grow out of one yearly marriage between persons of given ages, is saying very little to the purpose, and really, in some cases, seems incompatible with common sense. Thus, for instance, if $m = n$, and each equal to 6, $x = 20$, we shall find the number of marriages that will grow out of one yearly marriage, under these circumstances, to be by your theorem 25 nearly. But how is this possible? It is very clear there can be no more than 20 marriages in that time, arising from one marriage yearly; how the growth of the other five can be accounted for, I am entirely at a loss to conjecture. You, indeed, tell us, that 8 marriages will grow out of the yearly marriages in 10 years, 11.38 in 15 years, &c. By this, I suppose, you would be understood to mean,

that,

that, of the 10 marriages celebrated in that time, there will be left 8; that is, two of those marriages will be dissolved by the death of one or both of the parties engaged; likewise, the fifteen marriages will, in fifteen years, be reduced to 11, (the fraction being omitted) and so on proportionably to the maximum, which will take place at the expiration of 53 years; at that time you apprehend 19 couple will be found living: all this may possibly be fact; it may happen, that about 19 marriages will exist at the end of 53 years; but still, I say, these conclusions do not follow from any calculations introduced into this note. It is also to this note (L) you refer the reader for the investigation in note (A) where it is said, that if E represent any given expectation of life, $\frac{4E-x}{4E} \times px$ will be the number of persons alive at the end of $x$ years, arising from $p$, persons left annually as widows, (or added annually to a town or society,) at the age whose *expectation* is E, and consequently the *maximum* thereof always as $p$ E. For the application of this rule you propose the following example.

" At

"At the time of the commencement of the scheme, among the ministers and professors in *Scotland* for making provision for their widows, it was necessary, that a calculation should be made of the number of widows that would be upon the scheme at the end of every year, till they came to a *maximum*, on the supposition, that, (agreeably to what particuliar enquiry had shewn to have happened for many preceding years) 20 new widows would be left every year. In order to make this calculation, let 4 of the 20 widows be supposed to be under 32 years of age when left; and let 28 be supposed their mean age. Let the same number be left between 32 and 39, and let 35 be their mean age; between 39 and 47, and 43 their mean age; between 47 and 57, and 52 their mean age; between 57 and the extremity of life, and 63 their mean age. The number in life together, to which, in 10 years, 4 widows left annually at the age of 28 will grow, is by the rule (E being 29,) $\frac{116-10}{116} \times 40$, or 36.55.—The number alive, at the end of 20 years, will be $\frac{116-20}{116} \times 80$, or 66.2. At the end of 30 years, the number alive will be

be 89; of 40 years, 104.82; of 58 years 116.— Theſe numbers, found in the ſame way, for the 2d claſs, (E being 25.5) at the end of 10, 20, 30, 40, and 51 years, will be 36.7—64.31—84.7—97.25—102—for the 3d claſs, (E being 21.5) at the end of 10, 20, 30, 40, and 34 years, 35.34—61.4—78.13—85.6—86—For the 4th claſs, (E being 17) at the end of 10, 20, 30, and 34 years, 34.11—56.47—67—68— For the 5th claſs, (E being 11.5) at the end of 10, 20, and 23 years, 31.3—45.2—46—The whole number, therefore, conſiſting of all the claſſes, will come to a *maximum* nearly in 58 years; and the totals in life, at the end of 10, 20, 30, 40, 50, and 58 years, will be 173.37—293.58—364.83—401.67—418.

Theſe determinations ſuppoſe none to marry. In 10 years, from 1757 to 1767, I have been informed, that but 9 widows married. Let us then ſuppoſe, that one widow of the firſt claſs marries every year; and let all that marry be ſuppoſed to continue, one with another, 5 years in widow-hood before they marry. On theſe ſuppoſitions, the foregoing totals will, at the end of the ſame periods of years, be 169.23—282— 347.5—380.47—394.

These

These calculations are made from Mr. Demoivre's hypothesis. Had they been made from Dr. Halley's table, or any other of the tables I have given at the end of this work, except the *London* one, the results would have been very nearly the same.—It appears, that the probabilities of life, among these widows, are greater than those given by these tables. See the last essay, pages 263, 264, 266, 270, &c. The effect of this must be, to raise the *maximum* without sensibly increasing the numbers in life for the first 20 or 25 years: and its effect *may* be to raise the *maximums*, and at the same time even to deminish these numbers."

To this you add. "That twenty-five years have now elapsed since the commencement of this scheme; and the number of widows living every year have in fact, corresponded to the last numbers I have given, as nearly as could possibly be expected in an affair of this nature."

I must ingeniously own, that this extract bears a very plausible appearance of being true, and more especially as Dr. Price, whom I will not suppose could design any mental reservation

whatsoever

whatſoever in this affair, informs us the fact is, as above related, namely, that the number of widows living generally agreed with the numbers ariſing from the computations. Yet all this does not abſolutely prove the theorem to be univerſally true; and, certainly, if any particular caſe can be cited in which the theorem ſhall prove defective, I apprehend it may juſtly be ſuſpected to deviate from the truth in others; how far the numbers (although adventitious) in the example may be conducive to this coincidence with the obſervations abovementioned, I cannot pretend to determine; but it is very clear that if $x$ be taken equal to 4 E, the whole expreſſion will certainly vaniſh; and conſequently, be the number of perſons what they may, to be added to the ſociety annually, there will be none of them left alive at the expiration of $x$ years. This really ſeems abſurd, becauſe the propoſed number ($p$) is to be added alſo at the end of thoſe years, if I underſtand the example right. Previous to this, the ſame number was ſuppoſed to be likewiſe added at the end of $x—1$ years. Now (neglecting all the former additions) it ſeems very ſtrange, that, in this cir-

cumſtances, there ſhould not remain any of the added perſons alive at the expiration of $x$ years, be that number what it may, as the ſame number is conſtantly added annually until the expiration of $x-1$, if not of $x$ number of years.

It would extend this ſmall tract far beyond the deſigned limits, were I to enumerate and refute the miſtakes with which your treatiſe ſeems to abound; it may perhaps ſuffice to have already pointed out ſome of thoſe which are too conſpicuous to paſs unnoticed, otherwiſe I might advert to almoſt every note in the appendix, where ſcarce a page paſſeth without preſenting to the reader's view ſomething either ambiguous or defective; nay, even the very ſolutions in the body of the work, which are true, ſuch as the anſwers to queſtions 6th, 7th, and ſome few more, cannot be deduced from any inveſtigations to be found in the notes to which your readers are refered for that purpoſe; for inſtance, the ſolution to the 7th queſtion which is moſt indiſputably true, you aſſure us originates from note (C) yet it is very clear, I think, that no ſuch deduction can poſſibly be made from thence, where you tell us that $\frac{n-8}{\overline{n-7}\times r} \times \frac{1}{m-7} + \frac{n-9}{\overline{n-7}\times r^{2}} \times \frac{2}{m-7}$ &c. is the value of an annuity for a life ſeven years older than the expectant, after another life ſeven years older than the life whoſe complement is $m$, $n$ and $m$ being

being the refpective complements of the propofed ages, and $r=10.4$ as ufual. Your errors in this article feem to arife from a wrong interpretation of *Simpfon's doctrine of annuities* and his *Select Exercifes* with regard to the prefent value of all the poffible payments expounded by $\frac{1}{r^7}\times\frac{n-8}{nr}\times\frac{1}{m}+\frac{u-9}{nr^2}\times\frac{2}{m}+\frac{n-10}{nr^3}\times\frac{3}{m}$ &c. Mr. Simpfon's directions will not, I am very certain, lead to this determination. The firft probability, viz. that the life of the expectant fhall continue 8 years, and the other life 7, and fail in the 8th year, being $\frac{n-8}{n}\times\frac{1}{m}$ is the only juft conclufion in this note. The value of the 2d payment, at the end of the 9th year, viz. $\frac{n-9}{nr^9}\times\frac{m-7}{m}\times\overline{1-\frac{m-9}{m-7}}$ or $\frac{n-9}{nr^9}\times\frac{2}{m}$ and likewife of the other years appear (at leaft fo to me) very doubtful, and confequently all future reafoning upon this imperfect principle muft be alike defective; and yet you tell us, that this demonftration is, for the fake of more eafe and clearnefs, applied to the hypothefis of an equal decrement of life. It does not, however, depend upon it, but may be applied to any table of obfervations.

The note (E) as far as it relates to the intentended refutation of Demoivre's hypothefis, or rather to the eftablifhment of your own, refpecting your favourite correction to be applied

as

as a ſovereign remedy to all the common caſes concerning reverſionary annuities, is, as I have before obſerved, not only defective, but even abſurd in the latter part of it, where you conclude, that Mr. Demoivre's ſolution (a true one) being 9.919l. and your own (an untrue one) giving 10.316l. in anſwer to the ſame queſtion, it follows that becauſe 10.316 is to 9.919 in the ſame ratio with 104 to 100 or 26 to 25 (which by the bye is not quite exact) the rule given in the *Scholium*, page 34, contains the true and neceſſary correction to be uſed.

Note G ſeems juſt as inconſiſtent as the others. Here you tell us, that the rule in page 48, for reſolving the fourteenth queſtion, is nothing more than $\frac{1-a}{r}+\frac{1-b}{r^2}+\frac{1-c}{r^3}$ &c. $(t-1)$ $+\frac{1-P}{r^t}+\frac{1-P}{r^{t+1}}+\underline{\frac{1-P}{r^{t+2}}\text{ \&c. or its equal }\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}}$ &c. $(t)-\frac{a}{r}+\frac{b}{r^2}+\frac{c}{r^3}$ &c. $(t-1)+\frac{P}{r^t}+\frac{1-P}{r^t}\times\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}$ &c. expreſſed in words. The truth of this aſſertion, and of every other article againſt which I have objected in the courſe of this performance, I ſincerely wiſh you may make appear, if poſſible, in the next edition of your (then) celebrated treatiſe upon REVERSIONARY PAYMENTS, &c.

FINIS.



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Erratum.——Page 7. l. 21. for *her* deceaſe, read *his* deceaſe.